JOURNAL OF MULTIVARIATE ANALYSIS 4, 241–252 (1974)

Some Remarks on the Direct Limits of Measure Spaces

SERGE VASILACH

Département de Mathématiques, Université Laval, Québec 10, Canada

Communicated by M. M. Rao

The present paper is devoted to some remarks and alterations concerning our preceding articles [1–3].

1. THE PSEUDO σ-ALGEBRAS M AND $\mathfrak{M}$

Let I be a right preordered directed set; let $(E_\alpha)_{\alpha\in I}$ be a family of sets having I for set of indices. For each pair (α, β) of elements of I such that $\alpha \leqslant \beta$, let $f_{\beta\alpha}$ be an injective mapping of E_α into E_β. Suppose $(E_\alpha, f_{\beta\alpha})$ is a direct system of sets (cf. [1], Sec. 1, No 1).

Let $G = \bigcup_{\alpha\in I} E_\alpha x\{\alpha\}$ be the *sum* and $E = \varinjlim E_\alpha$ the direct limit (cf. [1], Section 1, Number 1), of the family $(E_\alpha)\alpha_{\in I}$. Let f be the canonical mapping of G on to E, and f_α the restriction of f to E_α. For each $\alpha \in I$, let $\mathfrak{M}_\alpha$ be a σ-algebra in E_α. Suppose the mapping $f_{\beta\alpha}$ is such that $\hat{f}_{\beta\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}_\beta$, whenever $\alpha \leqslant \beta$, where $\hat{f}_{\beta\alpha}$ is the *extension* injection of $f_{\beta\alpha}$ to the sets of subsets (cf. [1], Sec. 1, No. 3). Let $M = \bigcup_{\alpha\in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ be the *sum* and $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$, the direct limit of the family $(\mathfrak{M}_\alpha)_{\alpha\in I}$. Suppose $\hat{\Psi}_M$ is the canonical mapping of M into $\mathfrak{M}$ and $\hat{\Psi}_{\mathfrak{M}_\alpha}$ the restriction of $\hat{\Psi}_M$ to $\mathfrak{M}_\alpha$ (cf. [1], Sec. 2, No. 1). Under these conditions, we have:

PROPOSITION 1. *$\mathfrak{M}$ is endowed with an algebraic structure induced by the σ-algebras ($\mathfrak{M}_\alpha$ $\alpha \in I$) such that, for each $\alpha \in I$, $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ is a σ-algebra in $f_\alpha\langle E_\alpha\rangle$.*

Proof. For each $\alpha \in I$ let $\mathfrak{M}_\alpha{}^{\mathbb{N}}$ be the cartesian product $X_{n\in\mathbb{N}}\, \Omega_n$, where $\Omega_n = \mathfrak{M}_\alpha$ for each $n \in \mathbb{N} = \{0, 1, 2, 3,...\}$. Suppose $\hat{f}^{\mathbb{N}}_{\beta\alpha}$ is the extension of $\hat{f}_{\beta\alpha}$ to the product set $\mathfrak{M}_\alpha{}^{\mathbb{N}}$, and $\hat{h}_\alpha$ be the mapping of $\mathfrak{M}_\alpha{}^{\mathbb{N}}$ into $\mathfrak{M}_\alpha$ such that

$$\hat{h}_\alpha((X_\alpha{}^n)_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} X_\alpha{}^n \in \mathfrak{M}_\alpha . \tag{1}$$

Received April 20, 1973; revised August 27, 1973.
AMS Subject Classification: Primary 28A35.
Key words and phrases: Direct limits, measure spaces.

Then the family $(\hat{h}_\alpha)_{\alpha\in I}$ is a direct system of mapping, i.e., the diagram

$$\text{(I)} \qquad \begin{tikzcd} \mathfrak{M}_\alpha^{\mathbf{N}} \arrow[r, "\hat{h}_\alpha"] \arrow[d, "\hat{f}_{\beta\alpha}^N"'] & \mathfrak{M}_\alpha \arrow[d, "\hat{f}_{\beta\alpha}"] \\ \mathfrak{M}_\beta^{\mathbf{N}} \arrow[r, "\hat{h}_\beta"'] & \mathfrak{M}_\beta \end{tikzcd} \quad \text{is commutative.}$$

Indeed,

$$\hat{f}_{\beta\alpha}(\hat{h}_\alpha((X_\alpha{}^n)_{n\in\mathbf{N}})) = \hat{f}_{\beta\alpha}\left(\bigcup_{n\in\mathbf{N}} X_\alpha{}^n\right) = \bigcup_{n\in\mathbf{N}} f_{\beta\alpha}\langle X_\alpha{}^n\rangle.$$

On the other hand, we have

$$\hat{h}_\beta(\hat{f}_{\beta\alpha}^{\mathbf{N}}((X_\alpha{}^n)_{n\in\mathbf{N}})) = \hat{h}_\beta((\hat{f}_{\beta\alpha}(X_\alpha{}^n))_{n\in\mathbf{N}}) = \bigcup_{n\in\mathbf{N}} f_{\beta\alpha}\langle X_\alpha{}^n\rangle,$$

for each sequence $(X_\alpha{}^n)_{n\in\mathbf{N}}$ of elements of $\mathfrak{M}_\alpha$ and each $\alpha\in I$. Whence

$$\hat{h}_\beta \circ \hat{f}_{\beta\alpha}^{\mathbf{N}} = \hat{f}_{\beta\alpha} \circ \hat{h}_\alpha\,, \tag{2}$$

i.e., the diagram (I) is commutative, whenever $\alpha \leqslant \beta$. Therefore, (cf. [1], Sec. 1, No. 3, Prop. 4), there exists a unique mapping $\hat{h}$ of $\mathfrak{M}^{\mathbf{N}}$ into $\mathfrak{M}$, such that the diagram

$$\text{(II)} \qquad \begin{tikzcd} \mathfrak{M}_\alpha^{\mathbf{N}} \arrow[r, "\hat{h}_\alpha"] \arrow[d, "\hat{\Psi}_{\mathfrak{M}_\alpha}^{\mathbf{N}}"'] & \mathfrak{M}_\alpha \arrow[d, "\hat{\Psi}_{\mathfrak{M}_\alpha}"] \\ \mathfrak{M}^{\mathbf{N}} \arrow[r, "\hat{h}"'] & \mathfrak{M} \end{tikzcd} \quad \text{is commutative, i.e.,}$$

$$\hat{\Psi}_{\mathfrak{M}_\alpha} \circ \hat{h}_\alpha = \hat{h} \circ \hat{\Psi}_{\mathfrak{M}_\alpha}^{\mathbf{N}}\,. \tag{3}$$

But,

$$\hat{\Psi}_{\mathfrak{M}_\alpha}(\hat{h}_\alpha((X_\alpha{}^n)_{n\in N}))) = \hat{\Psi}_{\mathfrak{M}_\alpha}\left(\bigcup_{n\in\mathbf{N}} X_\alpha{}^n\right),$$

and (cf. [2], Sec. 1, Prop. 1),

$$\hat{\Psi}_{\mathfrak{M}_\alpha}\left(\bigcup_{n\in\mathbf{N}} X_\alpha{}^n\right) = f_\alpha\left\langle\bigcup_{n\in\mathbf{N}} X_\alpha{}^n\right\rangle = \bigcup_{n\in\mathbf{N}} f_\alpha\langle X_\alpha{}^n\rangle = \bigcup_{n\in\mathbf{N}} \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n),$$

whence,

$$\hat{\Psi}_{\mathfrak{M}_\alpha}(\hat{h}_\alpha((X_\alpha{}^n)_{n\in N})) = \bigcup_{n\in\mathbf{N}} \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n).$$

Therefore

$$\hat{h}(\hat{\Psi}^{\mathbf{N}}_{\mathfrak{M}_\alpha}((X_\alpha{}^n)_{n\in\mathbf{N}})) = \hat{h}((\hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n))_{n\in\mathbf{N}}) = \bigcup_{n\in\mathbf{N}} \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n)$$

for each $\alpha \in I$. More precisely:

If $\hat{h}_\alpha$ is the internal law of composition of the σ-algebra $\mathfrak{M}_\alpha$, which corresponds to the operation of countable union of elements of $\mathfrak{M}_\alpha$, i.e., if $\hat{h}_\alpha = \bigcup_{n\in\mathbf{N}}(\cdot)$ in E_α, then the restriction of $\hat{h}$ to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ is an internal law of composition in $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}$, which corresponds to the operation of countable union of elements of $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$, i.e., of subsets of $f_\alpha\langle E_\alpha\rangle \subset E$, by virtue of the relation $\hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha) = f_\alpha\langle X_\alpha\rangle$, for each $\alpha \in I$.

On the other hand, consider the mapping $\hat{\theta}_\alpha$ of $\mathfrak{M}_\alpha$ into $\mathfrak{M}_\alpha$ defined by

$$\hat{\theta}_\alpha(X_\alpha) = \theta_\alpha\langle X_\alpha\rangle = \complement_{E_\alpha} X_\alpha \in \mathfrak{M}_\alpha .$$

Then, the diagram

$$\text{(III)} \qquad \begin{array}{ccc} \mathfrak{M}_\alpha & \xrightarrow{\hat{\theta}_\alpha} & \mathfrak{M}_\alpha \\ {\scriptstyle \hat{f}_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle \hat{f}_{\beta\alpha}} \\ \mathfrak{M}_\beta & \xrightarrow[\hat{\theta}_\beta]{} & \mathfrak{M}_\beta \end{array} \qquad \text{commutes.}$$

Indeed,

$$\hat{f}_{\beta\alpha}(\hat{\theta}_\alpha(X_\alpha)) = \hat{f}_{\beta\alpha}\left(\complement_{E_\alpha} X_\alpha\right) = f_{\beta\alpha}\left\langle\complement_{E_\alpha} X_\alpha\right\rangle$$

and

$$\hat{\theta}_\beta(\hat{f}_{\beta\alpha}(X_\alpha)) = \hat{\theta}_\beta(f_{\beta\alpha}\langle X_\alpha\rangle) = \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle.$$

But, it is easy to show that $f_{\beta\alpha}$ injective whatever $\alpha \leqslant \beta$ implies

$$f_{\beta\alpha}\left\langle\complement_{E_\alpha} X_\alpha\right\rangle = \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle. \tag{4}$$

Indeed

$$E_\alpha = X_\alpha \cup \complement_{E_\alpha} X_\alpha \quad \text{and} \quad X_\alpha \cap \complement_{E_\alpha} X = \varnothing \Rightarrow f_{\beta\alpha}\langle E_\alpha\rangle = f_{\beta\alpha}\langle X_\alpha\rangle \cup f_{\beta\alpha}\left\langle\complement_{E_\alpha} X_\alpha\right\rangle$$

and

$$f_{\beta\alpha}\langle X_\alpha\rangle \cap f_{\beta\alpha}\left\langle\complement_{E_\alpha} X_\alpha\right\rangle = f_{\beta\alpha}\left\langle X_\alpha \cap \complement_{E_\alpha} X_\alpha\right\rangle$$

since $f_{\beta\alpha}$ is injective. On the other hand,

$$X_\alpha \cap \complement_{E_\alpha} X_\alpha = \varnothing \Rightarrow f_{\beta\alpha}\langle \varnothing \rangle = \varnothing, \quad \text{i.e.,}$$

$$f_{\beta\alpha}\langle X_\alpha\rangle \cap f_{\beta\alpha}\left\langle \complement_{E_\alpha} X_\alpha \right\rangle = \varnothing.$$

Likewise, we have

$$f_{\beta\alpha}\langle X_\alpha\rangle \cup \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle = f_{\beta\alpha}\langle E_\alpha\rangle,$$

$$f_{\beta\alpha}\langle X_\alpha\rangle \cap \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle = \varnothing,$$

whence

$$f_{\beta\alpha}\langle X_\alpha\rangle \cup f_{\beta\alpha}\left\langle \complement_{E_\alpha} X_\alpha \right\rangle = f_{\beta\alpha}\langle X_\alpha\rangle \cup \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle.$$

Then

$$y \in f_{\beta\alpha}\left\langle \complement_{E_\alpha} X_\alpha \right\rangle \Leftrightarrow y \notin f_{\beta\alpha}\langle X_\alpha\rangle$$

$$\Leftrightarrow y \in \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle \Leftrightarrow f_{\beta\alpha}\left\langle \complement_{E_\alpha} X_\alpha \right\rangle = \complement_{f_{\beta\alpha}\langle E_\alpha\rangle} f_{\beta\alpha}\langle X_\alpha\rangle. \qquad \text{Q.E.D.}$$

Then, it follows that the family $(\hat{\theta}_\alpha)_{\alpha\in I}$ is a direct system of mappings. Therefore (cf. [1], No. 3, Prop. 4), there exists a ***unique mapping*** $\hat{\theta}$ of $\mathfrak{M}$ into $\mathfrak{M}$ such that the diagram

(IV)
$$\begin{array}{ccc} \mathfrak{M}_\alpha & \xrightarrow{\hat{\theta}_\alpha} & \mathfrak{M}_\alpha \\ {\scriptstyle \hat{\Psi}_{\mathfrak{M}_\alpha}}\downarrow & & \downarrow{\scriptstyle \hat{\Psi}_{\mathfrak{M}_\alpha}} \\ \mathfrak{M} & \xrightarrow[\hat{\theta}]{} & \mathfrak{M} \end{array}$$
is commutative,

i.e.,

$$\hat{\Psi}_{\mathfrak{M}_\alpha} \circ \hat{\theta}_\alpha = \hat{\theta} \circ \hat{\Psi}_{\mathfrak{M}_\alpha}. \tag{5}$$

But

$$X_\alpha \in \mathfrak{M}_\alpha \Rightarrow \hat{\theta}_\alpha(X_\alpha) = \hat{\theta}_\alpha\langle X_\alpha\rangle = \complement_{E_\alpha} X_\alpha \in \mathfrak{M}_\alpha$$

$$\Rightarrow \hat{\Psi}_{\mathfrak{M}_\alpha}(\hat{\theta}_\alpha(X_\alpha)) = \hat{\Psi}_{\mathfrak{M}_\alpha}\left(\complement_{E_\alpha} X_\alpha\right) = f_\alpha\left\langle \complement_{E_\alpha} X_\alpha \right\rangle = \hat{\theta}\langle f_\alpha\langle X_\alpha\rangle\rangle = \hat{\theta}(\hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha)).$$

On the other hand, $f_{\beta\alpha}$ injective mapping whatever $\alpha \leqslant \beta \Rightarrow f_\alpha$ injective mapping $\forall \alpha \in I$ (cf. Bourbaki [4], Chap. III, p. 7, No. 6, Remark 1) $\Rightarrow$

$$f_\alpha \left\langle \complement_{E_\alpha} X_\alpha \right\rangle = \complement_{\hat{\Psi}_{\mathfrak{M}_\alpha}(E_\alpha)} \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha) = \complement_{f_\alpha\langle E_\alpha\rangle} f_\alpha\langle X_\alpha\rangle, \tag{6}$$

for each $\alpha \in I$, the proof of (6) being similar to the proof of (4) and in fact follows by the injectivity of f_α. Therefore, for each $\alpha \in I$, the restriction $\hat{\theta} \mid_{\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle}$ of $\hat{\theta}$ to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ is such that

$$\hat{\theta} \mid_{\hat{\Psi}_{\mathfrak{M}_\alpha}(E_\alpha)} = \theta \mid_{f_\alpha\langle E_\alpha\rangle} = \complement_{f_\alpha\langle E_\alpha\rangle} (\cdot) \qquad \text{in} \qquad \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle. \tag{7}$$

This means that for each $\alpha \in I$, the internal law of composition $\hat{\theta}_\alpha(\cdot)$ in $\mathfrak{M}_\alpha$, induces an internal law of composition in $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$, given by (6).

More precisely, for each $\alpha \in I$, the σ-algebra $\mathfrak{M}_\alpha$, induces a structure of a σ-algebra in $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$. This proves Proposition 1.

2. Alterations in [1]

The mapping $\hat{\Psi}_M$ is surjective, therefore (cf. [1], Sec. 1, No. 3, Prop. 3b), we have

$$\mathfrak{M} = \bigcup_{\alpha \in I} \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle.$$

By virtue of Proposition 1 of Section 1 above, the structure induced on E by the σ-algebras $(\mathfrak{M}_\alpha)_{\alpha\in I}$, is not a structure of σ-algebra in E, as specified in [1], Sec. 2, Theorem 1. More precisely, $\mathfrak{M}$ is endowed with a structure, which will be called a pseudo σ-algebra in E, in the sense of Proposition 1 (that is, for each $X \in \mathfrak{M}$, there exists an $\alpha \in I$ such that $X = f_\alpha\langle X_\alpha\rangle$, $X_\alpha \in \mathfrak{M}_\alpha$ and $\bigcup_{n\in\mathbb{N}} X_n \in \mathfrak{M} \Rightarrow \exists \alpha \in I$ such that $\bigcup_{n\in\mathbb{N}} X_n \in \mathfrak{M}_\alpha$ and $E \notin \mathfrak{M}$.

Likewise, Proposition 2 of [1], Sec. 1, is not true, i.e., M is not a σ-algebra in G, but a *pseudo σ-algebra*, in the sense that M is endowed with an algebraic structure such that for each $X \in M$ there exists a unique $\alpha \in I$ such that $X = X_\alpha \in \mathfrak{M}_\alpha$, and $(X_n)_{n\in\mathbb{N}} \subset M \Rightarrow \bigcup_{n\in\mathbb{N}} X_n \in M \Leftrightarrow \exists$ unique $\alpha \in I \curvearrowright \bigcup_{n\in\mathbb{N}} X_n \in \mathfrak{M}_\alpha \Leftrightarrow X_n \in \mathfrak{M}_\alpha$, $n \in \mathbb{N}$. ($\curvearrowright \Leftrightarrow$ such that)

M is called a *pseudo σ-algebra in G*, since $G \notin M$. Moreover (E, M) is not a measurable space, but *a pseudo-measurable space*: Under these conditions, Theorem 1 of Sec. 2, No. 1, must be altered as follows:

Theorem 1. *Suppose I is a (right) directed preordered set, $(E_\alpha, \Gamma_{\beta\alpha})$ is a direct system of sets relative to I, with respect to a family of injective mappings $(\Gamma_{\beta\alpha})$,*

$G = \bigcup_{\alpha\in I} E_\alpha \times \{\alpha\}$ is the sum and $E = \varinjlim E_\alpha = G/R$ is the direct limit of the family (E_α). Let $(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the direct system, extension of the direct system $(E_\alpha, \Gamma_{\beta\alpha})$; let

$$\hat{G} = \bigcup_{\alpha\in I} \mathfrak{P}(E_\alpha) \times \{\{\alpha\}\}$$

be the sum of the family $(\mathfrak{P}(E_\alpha))$; let $\hat{E} = \hat{G}/\hat{R}_G = \varinjlim(\mathfrak{P}(E_\alpha), \hat{\Gamma}_{\beta\alpha})$ be the direct limit of the family $(\mathfrak{P}(E_\alpha))$. For each $\alpha \in I$, let $\mathfrak{M}_\alpha$ be a σ-algebra in E_α, and suppose the mappings $\Gamma_{\beta\alpha}$ are such that, for each $X \in \mathfrak{M}_\alpha$, we have $\hat{\Gamma}_{\beta\alpha}(X) = \Gamma_{\beta\alpha}\langle X\rangle \in \mathfrak{M}_\beta$, whenever $\alpha \leqslant \beta$, i.e., *$\hat{\Gamma}_{\beta\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}_\beta$. Let $M = \bigcup_{\alpha\in I} \mathfrak{M}_\alpha \times \{\{\alpha\}\}$ be the sum and $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ the direct limit of the family $(\mathfrak{M}_\alpha)_{\alpha\in I}$. Suppose ϕ is the canonical mapping of G onto E; $\hat{\phi}$ is the extension of ϕ to the sets of subsets; $\hat{R}$ is the equivalence relation on $\mathfrak{P}(G)$ associated with $\hat{\phi}$; $\hat{\Psi}$ is the canonical mapping of $\mathfrak{P}(G)$ onto $\mathfrak{P}(G)/\hat{R}$, and g is the bijection* (cf. [5], No. 2, Theorem 1) of *$\mathfrak{P}(G/R)$ onto $\mathfrak{P}(G)/\hat{R}$. Under these conditions, $\mathfrak{M}$ is a pseudo-σ-algebra in $\hat{g}(E)$, and by identification of $\mathfrak{P}(G/R)$ with $\mathfrak{P}(G)/\hat{R}$, $\mathfrak{M}$ is a pseudo-σ-algebra in E.*

Likewise, the proof of Theorem 1, Section 3, No. 1, is the following. Suppose X, Y are two disjoint elements of $\mathfrak{M}$; then there exists a $\alpha \in I$, such that

$$X = \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha) = f_\alpha\langle X_\alpha\rangle \qquad \text{and} \qquad Y = \hat{\Psi}_{\mathfrak{M}_\alpha}(Y_\alpha) = f_\alpha\langle Y_\alpha\rangle,$$

whence

$f_\alpha\langle X_\alpha \cap Y_\alpha\rangle \subset f_\alpha\langle X_\alpha\rangle \cap f_\alpha\langle Y_\alpha\rangle = \varnothing \Rightarrow f_\alpha\langle X_\alpha \cap U_\alpha\rangle = \varnothing \Rightarrow X_\alpha \cap Y_\alpha = \varnothing$, since $f_\alpha\langle\varnothing\rangle = \varnothing$ and $X_\alpha \cap Y_\alpha \neq \varnothing \Rightarrow f_\alpha\langle X_\alpha \cap Y_\alpha\rangle \neq \varnothing$ (cf. [4], Chap. II, Part 3, No. 1). Then

$$\begin{aligned}\mu(X \cup Y) &= \mu(f_\alpha\langle X_\alpha\rangle \cup f_\alpha\langle Y_\alpha\rangle) = \mu(f_\alpha\langle X_\alpha \cup Y_\alpha\rangle)\\ &= \mu_\alpha(X_\alpha \cup Y_\alpha) = \mu_\alpha(X_\alpha) + \mu_\alpha(Y_\alpha) = \mu(f_\alpha\langle X_\alpha\rangle) + \mu(f_\alpha\langle Y_\alpha\rangle)\\ &= \mu(X) + \mu(Y).\end{aligned}$$

Therefore, μ is an additive mapping of $\mathfrak{M}$ into E.

3. The Countably Additive Mapping $\lambda = \varinjlim \lambda_\alpha$

Let $(\mathfrak{M}_\alpha . \hat{f}_{\beta\alpha})$ be the direct system of σ-algebras such as defined in [1], Section 2, Number 1, Theorem 1 (cf. Theorem 1, No. 2, above). Suppose $(f_{\beta\alpha})$ are injective mappings such that $\hat{f}_{\beta\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}_\beta$, whenever $\alpha \leqslant \beta$ in I.

Let $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ be the pseudo σ-algebra, direct limit of the family of σ-algebras $(\mathfrak{M}_\alpha)_{\alpha\in I}$.

Let $(F_\alpha, g_{\beta\alpha})$ be a direct system of additive Abelian groups, and $\mathfrak{F} = \varinjlim F_\alpha$ the Abelian group, direct limit of the family $(F_\alpha)_{\alpha\in I}$. Suppose, moreover, that $F_\alpha = F$, $\forall \alpha \in I$, where F is a complete additive Abelian group. Let (cf. [1], Sec. 4, No. 2) λ_α be a countably additive mapping of $\mathfrak{M}_\alpha$ into F, for each $\alpha \in I$; then, if $\lambda = \varinjlim \lambda_\alpha$ is the direct limit of the family $(\lambda_\alpha)_{\alpha\in I}$, we have $\lambda_\alpha = \lambda \circ \hat{\Psi}_{\mathfrak{M}_\alpha}$, and λ is an additive mapping of the pseudo σ-algebra $\mathfrak{M}$ into F. (cf. Theorem 1, Sec. 3, No. 1). Moreover, the restriction of λ to the σ-algebra $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ is countably additive mapping of $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ into F, for each $\alpha \in I$.

Indeed; let $(X_n)_{n\in\mathbb{N}}$ be a sequence of disjoint elements of $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$. Then

$$X_n = \hat{\Psi}_{\mathfrak{M}_\alpha}\langle X_\alpha{}^n\rangle, \qquad X_\alpha{}^n \in \mathfrak{M}_\alpha,$$

$$\forall n \in \mathbb{N} \Rightarrow \lambda\left(\bigcup_{n\in\mathbb{N}} X_n\right) = \lambda\left(\bigcup_{n\in\mathbb{N}} \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n)\right)$$

$$= \lambda\left(\hat{\Psi}_{\mathfrak{M}_\alpha}\left(\bigcup_{n\in\mathbb{N}} X_\alpha{}^n\right)\right) = \lambda_\alpha\left(\bigcup_{n\in\mathbb{N}} X_\alpha{}^n\right).$$

But

$$X_n \bigcup_{n\neq m} X_m \Rightarrow \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n) \cap \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^m) = \varnothing$$

$$\Rightarrow \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n \cap X_\alpha{}^m) \subset \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n) \cap \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^m) = \varnothing$$

$$\Rightarrow X_\alpha{}^n \cap X_\alpha{}^m = \varnothing \qquad \text{(cf. no 2, above)}$$

$$\Rightarrow \lambda_\alpha\left(\bigcup_{n\in\mathbb{N}} X_\alpha{}^n\right) = \sum_{n\in\mathbb{N}} \lambda_\alpha(X_\alpha{}^n)$$

$$= \sum_{n\in\mathbb{N}} \lambda(\hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha{}^n)) = \sum_{n\in\mathbb{N}} \lambda(X_n). \qquad \text{Q.E.D.}$$

Therefore, the Theorem 2 of [1], Section 4, Number 3, must be altered as follows:

THEOREM 2. *Under the hypothesis of Theorem* 1, *Section* 2, *and Theorem* 1, *Section* 3, *let* $(\lambda_\alpha)_{\alpha\in I}$ *be a direct system of measures, with values in a complete additive abelian group F. More precisely, let* $\lambda_\alpha\colon \mathfrak{M}_\alpha \to F$ *be a measure in* $\mathfrak{M}_\alpha$, *with values in F, for each* $\alpha \in I$. *Let* $\lambda = \varinjlim \lambda_\alpha$ *be the direct limit of the direct system of measures* $(\lambda_\alpha)_{\alpha\in I}$.

Then, the restriction of λ *to the* σ*-algebra* $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ *is a measure on* $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ *with values in F, for each* $\alpha \in I$.

Likewise, Theorem 3 of [1], Section 4, Number 3, must be replaced by the following.

THEOREM 3. $(E, \mathfrak{M}, \lambda) = (\varinjlim E, \varinjlim \mathfrak{M}_\alpha, \varinjlim \lambda_\alpha)$ *is a pseudo σ-measure space, direct limit of the family* $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ *of measure spaces*, in the sense that the restriction of λ to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$ is a measure in $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$. In particular, if $(E_\alpha, \mathfrak{M}_\alpha, p_\alpha)_{\alpha \in I}$ is a direct system of probability spaces, then the following holds.

THEOREM 4. (which replaces Theorem 4 of [3], Sec. 4, No. 3). $(E, \mathfrak{M}, p) = (\varinjlim E_\alpha, \varinjlim \mathfrak{M}_\alpha, \varinjlim p_\alpha)$ *is a pseudo probability spaces, in the sense that the restriction of* $p = \varinjlim p_\alpha$ to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$ *is a probability in* $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$, *for each* $\alpha \in I$:

Typographical linguistic points in [1]. Everywhere replace "countable additive" by "countably additive" M (respectively $\mathfrak{M}$) σ-algebra by M (respectively $\mathfrak{M}$) pseudo-σ-algebra "(G, M) measurable space" by "(G, M) pseudo measurable space."

Beginning Section 2, replace everywhere "measurable space" by "pseudo measurable space," "correspondence $\Gamma_{\beta\alpha}$" by "injective mapping $\Gamma_{\beta\alpha}$," "a unique α" by "a α." "λ is a countable additive mapping" by "the restriction of λ to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$ is a countably additive mapping of $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$ into F."

The proof given in Section 4, Number 3, must be replaced by the proof of Number 3, above.

4. ALTERATIONS IN [2]

First of all, replace: "mappings $f_{\beta\alpha}$" by injective mappings $f_{\beta\alpha}$. M is a σ-algebra in G (respectively, $\mathfrak{M}$ is a σ-algebra in E) by M is a *pseudo σ-algebra* in G (respectively, $\mathfrak{M}$ *is a pseudo σ-algebra in E*). In Proposition 2, Section 1, the assertion (b) must be altered as follows:

(b) *The restriction of u to* $f_\alpha\langle E_\alpha \rangle$ *is a* $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle, \mathfrak{N})$ *measurable function, for each* $\alpha \in I$.

Proof of (b). u_α: $(\mathfrak{M}_\alpha, \mathfrak{N})$ measurable function: $\forall \alpha \in I \Rightarrow \forall Y \in \mathfrak{N}$, we have $u_\alpha^{-1}\langle Y \rangle = X_\alpha \in \mathfrak{M}_\alpha$. Then, from assertion (a) of Proposition 2, Section 1, Number 2, of [2], we have $u_\alpha^{-1}\langle Y \rangle = f_\alpha^{-1}\langle u^{-1}\langle Y \rangle \rangle = X_\alpha$. But, by Lemma 1 (loc. cit.) we get $f_\alpha\langle X_\alpha \rangle = u^{-1}\langle Y \rangle$. But $f_\alpha\langle X_\alpha \rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle$; therefore $u^{-1}\langle Y \rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle \mathfrak{M}_\alpha \rangle \subset \mathfrak{M}$. Q.E.D.

In Proposition 3 of [2], Section 1, Number 2, the assertion: "Let $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ (respectively, $\mathfrak{N} = \varinjlim \mathfrak{N}_\alpha$) *be the direct limit σ-algebra of the family* $(\mathfrak{M}_\alpha)$ (respectively, $\mathfrak{N}_\alpha$). *Under these conditions, the direct limit mapping* $u = \varinjlim u_\alpha$, *of E into F is a* $(\mathfrak{M}, \mathfrak{N})$ *measurable function*" will be altered as follows: "*Let* $\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ (respectively, $\mathfrak{N} = \varinjlim \mathfrak{N}_\alpha$) *be the pseudo-σ-algebra of the family* $(\mathfrak{M}_\alpha)$ (respectively, $(\mathfrak{N}_\alpha)$). *Under these condition, the restriction of the direct limit* *mapping* $u = \varinjlim u_\alpha$ to $f_\alpha\langle E_\alpha\rangle$ is $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \hat{\theta}_{\mathfrak{N}_\alpha}\langle\mathfrak{N}_\alpha\rangle$ *measurable function, for each* $\alpha \in I$.

Proof. $u = \varinjlim u_\alpha \Rightarrow u \circ f_\alpha = g_\alpha \circ u_\alpha$ for each $\alpha \in I$ and $\forall Y \in \mathfrak{N}$, $\Rightarrow$ there exists a $\alpha \in I$ such that $Y \in \hat{\theta}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$, where $\hat{\theta}_{\mathfrak{N}_\alpha}$ is the restriction to $\mathfrak{N}_\alpha$ of the canonical mapping $\hat{\theta}$ of the pseudo σ-algebra $N = \bigcup_{\alpha\in I} \mathfrak{N}_\alpha \times \{\{\alpha\}\}$ into the pseudo σ-algebra $\mathfrak{N} = \bigcup_{\alpha\in I} \hat{\theta}_{\mathfrak{N}_\alpha}\langle\mathfrak{N}_\alpha\rangle$. But

$$Y = \hat{\theta}_{\mathfrak{N}_\alpha}(Y_\alpha) \in \hat{\theta}_{\mathfrak{N}_\alpha}\langle\mathfrak{N}_\alpha\rangle \Rightarrow Y = g_\alpha\langle Y_\alpha\rangle \qquad \text{for} \quad y_\alpha \in \mathfrak{N}_\alpha,$$

by virtue of Lemma 1 of [2], loc. cit.

On the other hand, u_α: $(\mathfrak{M}_\alpha, \mathfrak{N}_\alpha)$ measurable function $\Rightarrow u_\alpha^{-1}\langle Y_\alpha\rangle = X_\alpha \in \mathfrak{M}_\alpha \Rightarrow Y_\alpha = u_\alpha\langle X_\alpha\rangle$ (cf. Lemma 1, loc. cit.) $\Rightarrow g_\alpha\langle Y_\alpha\rangle = Y$. But $g_\alpha \circ u_\alpha = u \circ f_\alpha \Rightarrow X_\alpha = u_\alpha^{-1} \circ g_\alpha^{-1}(Y) = f_\alpha^{-1} \circ u^{-1}(Y) \Rightarrow u^{-1}(Y) = f_\alpha(X_\alpha)$ by injectivity of f_α. However, we have $f_\alpha\langle X_\alpha\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}$, whence $u^{-1}\langle Y\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M}$. Therefore u is $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \hat{\theta}_{\mathfrak{N}_\alpha}\langle\mathfrak{N}_\alpha\rangle)$ measurable function, for each $\alpha \in I$. Q.E.D.

In Section 2, Number 1, of [2], $(E, \mathfrak{M}, \lambda)$ is the pseudo measure space, direct limit of the system $(E_\alpha, \mathfrak{M}_\alpha, \lambda_\alpha)$ of measure spaces, $u = \varinjlim u_\alpha$ is $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \mathscr{B}_+)$ measurable function for each $\alpha \in I$, and the integral of u for each $X \in \mathfrak{M}$, with respect to the measure defined by the restriction of the additive mapping λ to the σ-algebra $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$, is given by

$$I(u) = \int_X u \, d\lambda = \sup \int_X s \, d\lambda, \quad \text{for} \quad X = \hat{\Psi}_{\mathfrak{M}_\alpha}(X_\alpha) = f_\alpha\langle X_\alpha\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}(\mathfrak{M}_\alpha) \subset \mathfrak{M},$$

where the supremum is taken over all $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \mathscr{B}_+)$ measurable simple functions s of $f_\alpha\langle E_\alpha\rangle$ into $\mathbb{R}_+$, such that $0 \leqslant s \leqslant u$. That is, $s = \sum_{f\in J} a_j \phi_{X_j}$, $X_j \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$, $a_j \in \overline{\mathbb{R}}_+$. Thus

$$\int_X s \, d\lambda = \sum_j a_j \lambda(X \cap X_j) = \sum_{j\in J} a_j \lambda(f_\alpha\langle X_\alpha\rangle \cap f_\alpha\langle X_\alpha{}^j\rangle)$$

$$= \sum_{f\in J} a_j \lambda(f_\alpha\langle X_\alpha \cap X_\alpha{}^j\rangle) = \int_{X_\alpha} s_\alpha \, d\lambda_\alpha,$$

precisely:

In Sec. 1, No. 2, Proposition 3; replace: "$f_{\beta\alpha}$ (respectively, $g_{\beta\alpha}$)" by "the injective mapping $f_{\beta\alpha}$ (respectively, $g_{\beta\alpha}$)." "direct limit mapping $u = \varinjlim u_\alpha$ is $(\mathfrak{M}, \mathfrak{N})$ measurable function," by "the restriction of the direct limit mapping $u = \varinjlim u_\alpha$ to $f_\alpha\langle E_\alpha\rangle$ is a $(\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \hat{\theta}_{\mathfrak{N}_\alpha}\langle\mathfrak{N}_\alpha\rangle)$ *measurable* function, for each $\alpha \in I$. For this reason we shall say that $u = \varinjlim u_\alpha$ is a $(\mathfrak{M}, \mathfrak{N})$ *pseudo-measurable function.*"

Section 2, Number 1, replace: "limit measure space" by "limit pseudo-measure space," "positive measure $\lambda = \varinjlim \lambda_\alpha$" by "positive pseudo-measure $\lambda = \varinjlim \lambda_\alpha$," "measurable simple function s" by "pseudo-measurable simple function s."

In Theorem 1, Section 2, replace: "the measure on $\mathfrak{M}$" by "the pseudo-measure in $\mathfrak{M}$," "limit measure," by "limit pseudo-measure," "measurable function u" by "pseudo-measurable function u," "measure λ" by "pseudo-measure λ," "a unique $\alpha \in I$" by "a $\alpha \in I$."

Sec. 2, no 3; replace: "$(E, \mathfrak{M}, P)$... probability space" by: (...)... pseudo-probability space," "probability" by: "pseudo-probability". "a unique $\alpha \in I$" by: "a $\alpha \in I$". "$u = \varinjlim u_\alpha$ ... real random variable" by: "$u = \ldots$ real pseudo-random variable". "unique" should be deleted. Page 25, Reference [2], replace "Addison-Wesley" by: "Hermann, Paris."

5. *Alterations in* [3]. Everywhere replace "$f_{\beta\alpha}$ is a mapping" by "$f_{\beta\alpha}$... injective mapping", "... $\mathfrak{M}$: σ-algebra in E" by "...: pseudo-σ-algebra in E," "P is a countably additive mapping of the σ-algebra $\mathfrak{M}$ into $=[0, 1]$" by "the restriction of P to $f_\alpha\langle E_\alpha\rangle$ is countably additive mapping of the σ-algebra $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ into $F = [0, 1]$, for each $\alpha \in I$." Equation (6) must be written

$$0 \leqslant P(X) \leqslant 1, \qquad \forall X = f_\alpha\langle X_\alpha\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle \subset \mathfrak{M} \tag{6}$$

Page 5, lines 2–12 should be deleted. "$(E, \mathfrak{M}, P) = (\varinjlim E, \varinjlim \mathfrak{M}_\alpha, \lim P_\alpha)$: probability space" should be replaced by "(...) (...): pseudo-probability space."

Section 1, Number 2, Proposition 1, must be altered as follows:

PROPOSITION 1. *Let $(E, \mathfrak{M}, P)$ be the pseudo-probability space, direct limit of a direct system $(E_\alpha, \mathfrak{M}_\alpha, P_\alpha)$ of probability spaces.*

(a) *If $(E, \mathfrak{M}, P)$ is a complete pseudo-probability space (i.e., if for each $\alpha \in I$, the restriction of P to $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ is a complete probability space), then for each P-negligible subset N of $\mathfrak{M}$ there exists a $\alpha \in I$ such that $N = f_\alpha\langle N_\alpha\rangle$, $N_\alpha \in \mathfrak{M}_\alpha$, and N_α is a P_α-negligible subset of E_α.*

(b) *Conversely, if $(E_\alpha, \mathfrak{M}_\alpha, P_\alpha)$ is a direct system of complete probability spaces, then $(E, \mathfrak{M}, P)$ is a complete pseudo-probability space.*

Proof of (a). If $(E, \mathfrak{M}, P)$ is a complete pseudo probability space, then for each $\alpha \in I$, $\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ contains all the $P|_{\tilde{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle}$ negligible subsets of $f_\alpha\langle E_\alpha\rangle$. Let Ω_α be the set of all $P|_{\hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle}$ negligible subsets of $f_\alpha\langle E_\alpha\rangle$; we have $\Omega_\alpha \subset \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ and $\forall N \in \Omega_\alpha$, $\exists A \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$ such that $N \subset A$ and $P(A) = 0$. On the other hand, we have $N = \hat{\Psi}_{\mathfrak{M}_\alpha}(N_\alpha)$ for $N_\alpha \in \mathfrak{M}_\alpha$, and $A = \hat{\Psi}_{\mathfrak{M}_\alpha}(A_\alpha)$, $A_\alpha \in \mathfrak{M}_\alpha$. But, according to ([2], Sec. 1, Prop. 1), we have $N = f_\alpha\langle N_\alpha\rangle$ and $A = f_\alpha\langle A_\alpha\rangle$. Thus ([2], Section 1, Lemma 1) $N_\alpha = f_\alpha^{-1}\langle N\rangle$ and $A_\alpha = f_\alpha^{-1}\langle A\rangle$. Then $N \subset A \Rightarrow$

$N_\alpha \subset A_\alpha$, and $P(A) = 0 \Rightarrow P(f_\alpha\langle A_\alpha\rangle) = P_\alpha(A_\alpha) = 0 \Rightarrow N_\alpha$ *is a* P_α*-negligible subset of* E.

Proof of (b). Suppose $(E_\alpha, \mathfrak{M}_\alpha, P_\alpha)$ is a direct system of complete probability spaces. Then, if N is a P-negligible subset of E, there exists an element $A \in \mathfrak{M}$ such that $N \subset A$ and $P(A) = 0$. Moreover, then, there exists a $\alpha \in I$ such that $A = f_\alpha\langle A_\alpha\rangle$, $A_\alpha \in \mathfrak{M}_\alpha$, whence

$$0 = P(A) = P(f_\alpha\langle A_\alpha\rangle) = P_\alpha(A_\alpha).$$

On the other hand, $N \subset f_\alpha\langle A_\alpha\rangle \Rightarrow \exists N_\alpha \subset A_\alpha$, $N = f_\alpha\langle N_\alpha\rangle \subset f_\alpha\langle A_\alpha\rangle = A$, whence (cf. [2], Section 1, Lemma 1):

$$f_\alpha^{-1}\langle N\rangle = N_\alpha \subset f_\alpha^{-1}\langle A\rangle = A_\alpha \Rightarrow N_\alpha$$

is P_α-negligible subset of $E_\alpha \Rightarrow N_\alpha \in \mathfrak{M}_\alpha$, since $(E_\alpha, \mathfrak{M}_\alpha, P_\alpha)$ is a complete probability space $\Rightarrow N = f_\alpha\langle N_\alpha\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle$. Then we say that $(E, \mathfrak{M}, P)$ is a *complete pseudo-probability space.* Q.E.D.

In Section 1, Number 3, replace "completion of probability spaces" by "completion of pseudo probability spaces."

In Proposition 2: After "completion of $(E, \mathfrak{M}, P)$" *add* "in the sense that $(f_\alpha\langle E_\alpha\rangle, \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, \bar{P})$ is the completion of $(f_\alpha\langle E_\alpha\rangle, \Psi_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle, P)$, for each $\alpha \in I$."

On page 8 line 18, and p. 9 line 2: "unique" should be deleted. On page 9 line 3, replace "$N = f_\alpha\langle X_\alpha\rangle$" by "$N = f_\alpha\langle N_\alpha\rangle$," on page 9 line 13, replace:

$$\text{“}f_\alpha\langle N_\alpha\rangle \in \overline{\mathfrak{M}}\text{”} \qquad \text{by} \qquad \text{“}f_\alpha\langle N_\alpha\rangle \in \hat{\Psi}_{\mathfrak{M}_\alpha}\langle\mathfrak{M}_\alpha\rangle\text{”}.$$

In Section 1, Number 4, Page 10, line 1, replace: "probability" by "pseudo-probability." Page 11, line 17, replace: "conditional probability" by "conditional pseudo probability." Page 11, line 4, "unique" should be deleted. Page 11, line 18, and line 21, replace "conditional probability" by "conditional pseudo probability." Page 12, line 1: "unique" should be deleted. Page 12, line 6, replace: "conditional probability" by "conditional pseudo probability." Page 14, line 7, "unique" should be deleted.

In Section 1, Number 5, replace

Page 15, line 5, "probability space" by "pseudo probability space." Page 15, line 13, "r.r.v." by "pseudo r.r.v." Page 15, line 5, "probability" by "pseudo probability." Page 16, line 19, "to" by "two." Page 17, line 19, "probability" by "pseudo probability." Page 18, line 5, "r.r.v." by "pseudo r.r.v." Page 18, line 6, "r.r.v." by "pseudo r.r.v." Page 18, line 15, "r.r.v." by "pseudo r.r.v." Page 18, line 18, "$P(\{| u_\alpha^n - u_\alpha | > \epsilon\})$" by "$P(\{| \varinjlim u_\alpha^n - \varinjlim u_\alpha | > \epsilon\})$."

In Section 2, replace everywhere: "$(E, \mathfrak{M}, P)$ measure space" by "$(E, \mathfrak{M}, P)$ pseudo-measure space." "$\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ (respectively, $\mathfrak{N} = \lim \mathfrak{N}_\alpha$): σ-algebra" by "$\mathfrak{M} = \varinjlim \mathfrak{M}_\alpha$ (respectively, $\mathfrak{N} = \varinjlim \mathfrak{N}_\alpha$): pseudo-$\sigma$-algebra." $\varinjlim(E_\alpha, \mathfrak{N}_\alpha, \lambda_{\mathfrak{N}_\alpha})$: measure subspace" by $\varinjlim(E_\alpha, \mathfrak{N}_\alpha, \lambda_{\mathfrak{N}_\alpha})$: pseudo-measure subspace." "$(E, \mathfrak{N}, \lambda_{\mathfrak{N}})$ ... measure space" by "$(E, \mathfrak{N}, \lambda_{\mathfrak{N}})$... pseudo-measure space." "$(E, \mathfrak{N}, P_{\mathfrak{N}}) =$ ... is a probability space" by "$(E, \mathfrak{N}, P_{\mathfrak{N}}) =$ is a pseudo probability space." "mappings $f_{\beta\alpha}$" by "injective mappings $f_{\beta\alpha}$". "$u = \varinjlim u_\alpha$ ... positive r.r.v." by "$u = \varinjlim u_\alpha$ ... positive pseudo r.r.v."

Remark. It follows that the pseudo-structures of direct limits *give the same results as if these should be veritable structures.*

References

[1] Vasilach, S. (1971). Direct limits of measure spaces. *J. Multivar. Anal.* **1**, No. 4, 374–411.

[2] Vasilach, S. (1973). Integration of direct limits of real random variables. *J. Multivar. Anal.* **3**, No. 1, 17–25.

[3] Vasilach, S. (1973). A calculus of direct limits of probability space. *J. Multivar. Anal.* **3**, No. 1, 1–16.

[4] Bourbaki, N. (1970). *Théorie des Ensembles.* Hermann Editeur, Paris.